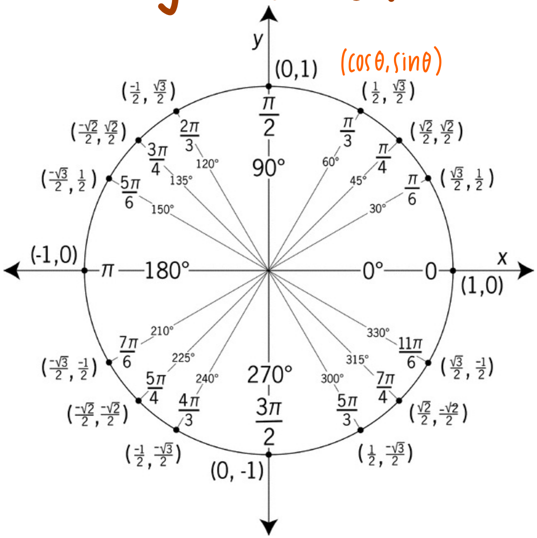
trig review



$\tan\theta = \frac{\sin\theta}{\cos\theta}$ $\csc\theta = \frac{1}{\sin\theta}$ $\sin^2\theta + \cos^2\theta = 1$

$\cot\theta = \frac{\cos\theta}{\sin\theta}$ $\sec\theta = \frac{1}{\cos\theta}$ SOHCAHTOA

inverses ($f^{-1}(x)$)

find inverse
1. switch x & y
2. solve for y
3. sub $f^{-1}(x)$ for y

$f(x)$ has inverse if $f(x)$ is one-to-one → horizontal line test

inverse check

$f(f^{-1}(x)) = x$ & $f^{-1}(f(x)) = x$

exponential

inverses!

$b^{\log_b(x)} = x$

$\log_b(b^x) = x$

logarithmic

$f(x) = b^x$ (base b) ; $b > 0$ & $b \neq 1$

domain: $(-\infty, \infty)$ range: $(0, \infty)$

$f(x) = \log_b(x)$ (base b)

laws

$a^{x+y} = a^x a^y$

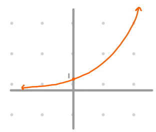
$a^{x-y} = \frac{a^x}{a^y}$

$(a^x)^y = a^{xy}$

$(ab)^x = a^x b^x$

natural exponential

$f(x) = e^x$



laws

$\log_b(xy) = \log_b(x) + \log_b(y)$

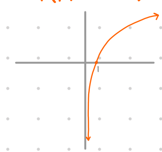
$\log_b(x/y) = \log_b(x) - \log_b(y)$

$\log_b(x^r) = r\log_b(x)$

$\log_b(b) = 1$

natural logarithm

$f(x) = \ln(x)$

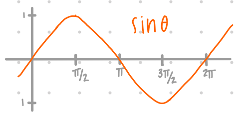


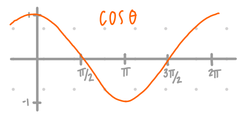
inverse trig

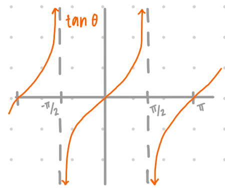
input: angle output: number

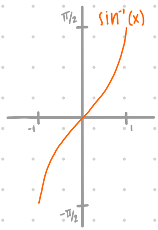
i.e. For $\cos^{-1}(1/2) = ?$, ask, "what angle within range gives $\cos\theta = 1/2$?"

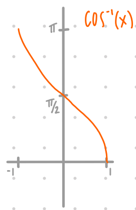
function	range
$y = \sin^{-1}(\theta)$	$y \in [-\pi/2, \pi/2]$
$y = \cos^{-1}(\theta)$	$y \in [0, \pi]$
$y = \tan^{-1}(\theta)$	$y \in (-\pi/2, \pi/2)$
$y = \csc^{-1}(\theta)$	$y \in (0, \pi/2] \cup (\pi, 3\pi/2]$
$y = \sec^{-1}(\theta)$	$y \in [0, \pi/2) \cup [\pi, 3\pi/2)$
$y = \cot^{-1}(\theta)$	$y \in (0, \pi)$

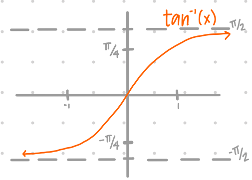












Pre-Calc

limits

δ-ε definition

For every $\varepsilon>0$, there is a $\delta>0$ such that if $0<|x-a|<\delta$, then $|f(x)-L|<\varepsilon$

prove $\lim_{x\to a} f(x)=L$

1. scratch: start with $|f(x)-L|<\varepsilon$ and get $|f(x)-L|$ to look like $|x-a|$ to find δ
2. proof:
   - "Given $\varepsilon>0$, let $\delta=$ ___"
   - "If $0<|x-a|<\delta$, then scratch work to show $|f(x)-L|<\varepsilon$"
   - "By definition of a limit, $\lim_{x\to a} f(x)=L$." ■

computing limits

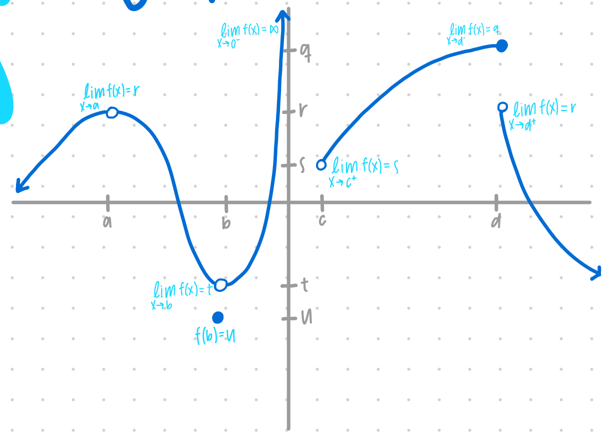
direct sub (plug in a) → zero in denominator?

- no → done
- yes → zero in numerator?
  - yes →
    - algebraically rearrange
    - factor
    - multiply by conjugate
    - squeeze theorem
    - possible limit at infinity
  - no → infinite limit
    - ∞ $(+/+, -/-)$
    - $-\infty$ $(+/-, -/+)$

conjugate of $\sqrt{a}\pm b$ is $\sqrt{a}\mp b$

$$\lim_{x\to a} f(x)=L \iff \underbrace{\lim_{x\to a^-} f(x)}_{\text{left-sided}} = L = \underbrace{\lim_{x\to a^+} f(x)}_{\text{right-sided}}$$

graphically



squeeze theorem

if $f(x)\le g(x)\le h(x)$ and $\lim_{x\to a} f(x)=\lim_{x\to a} h(x)=L$, then $\lim_{x\to a} g(x)=L$

- $-1\le \sin(\theta)\le 1$
- $-1\le \cos(\theta)\le 1$

infinite limits $\lim_{x\to a} f(x)=\pm\infty$

- look at left & right sided limits
- speculate or plug in appropriate number to figure out **sign** of numerator & denominator
- $\infty = \frac{+}{+}$ or $\frac{-}{-}$ $-\infty = \frac{+}{-}$ or $\frac{-}{+}$
- if $\lim_{x\to a^-} f(x)=\lim_{x\to a^+} f(x)=\infty$, then $\lim_{x\to a} f(x)=\pm\infty$

vertical asymptote:
- solve denominator=0
- verify with EITHER $\lim_{x\to a^-} f(x)=\pm\infty$ $\lim_{x\to a^+} f(x)=\pm\infty$

limits at infinity $\lim_{x\to\pm\infty} f(x)=L$

laws

- $\lim_{x\to\pm\infty} \frac{1}{x^n}=0$
- if n is even, $\lim_{x\to\infty} x^n=\infty$ & $\lim_{x\to-\infty} x^n=\infty$
- if n is odd, $\lim_{x\to\infty} x^n=\infty$ & $\lim_{x\to-\infty} x^n=-\infty$

rational functions

1. choose highest power in denominator
2. divide terms by highest power
3. take limit of each term

• important! •

$x\to\infty$: $\sqrt{x^2}=|x|=x$

$x\to-\infty$: $\sqrt{x^2}=|x|=-x$

horizontal asymptote: $y=b$ if $\lim_{x\to\infty} f(x)=b$ or $\lim_{x\to-\infty} f(x)=b$

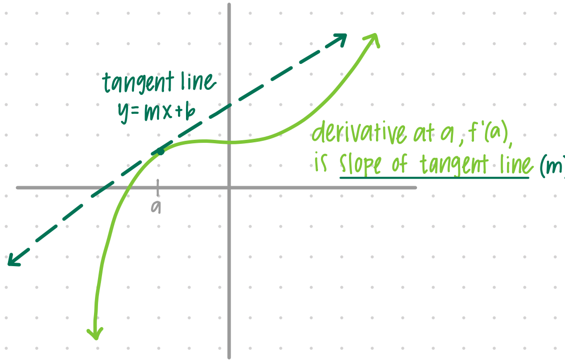
slant asymptote: degree of numerator = degree of denominator +1
- use long polynomial division

derivatives

- slope of tangent line
- at a point $x=a$: $f'(a)=\lim\limits_{x\to a}\dfrac{f(x)-f(a)}{x-a}$

  OR

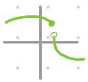
  $f'(a)=\lim\limits_{h\to 0}\dfrac{f(a+h)-f(a)}{h}$ (number)
- as a function: $f'(x)=\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$ (function)



differentiability & continuity

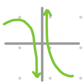
discontinuities

jump



removable

infinite



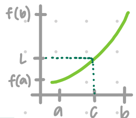
oscillating

continuous at a:

- $f(a)$ defined
- $\lim\limits_{x\to a} f(x)=a$ exists
- $\lim\limits_{x\to a} f(x)=f(a)$

IVT

If f is continuos on [a,b] and $f(a)<L<f(b)$, then $a<c<b$ where $f(c)=L$

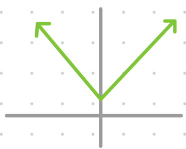


differentiability $\Rightarrow$ continuity
continuity $\not\Rightarrow$ differentiability

non-differentiable

- discontinuities
- corners & cusps
- vertical tangents

example: corner



graphs of f(x) and f'(x)

f(x)	f'(x)
increasing	positive (above x-axis)
decreasing	negative (below x-axis)
horizontal tangent	root (crosses x-axis)
not differentiable at a	f'(a) undefined

tangent line

find equation of tangent line

given f(x) and $x_1=a$...

1. find slope of tangent line, m_{tan}, at point $x_1=a$ by plugging in a into derivative

   $m_{tan\,@a}=f'(a)$
2. find y_1 value by plugging in $x_1=a$ into original function, f(x)

   $y_1=f(a)$
3. solve point-slope equation for y:

   $y-y_1=m_{tan}(x-a)$

normal line

find equation of normal line

given f(x) and $x_1=a$...

1. find slope of tangent line, m_{tan}, at point $x_1=a$ by plugging in a into derivative

   $m_{tan\,@a}=f'(a)$
2. find slope of normal line, m_{norm}, at $x_1=a$, by the negative reciprocal of m_{tan}.

   $m_{norm\,@a}=\dfrac{-1}{m_{tan}}$
3. find y_1 value by plugging in $x_1=a$ into original function, f(x)

   $y_1=f(a)$
4. solve point-slope equation for y:

   $y-y_1=m_{norm}(x-a)$

taking derivatives

- recall: $\sqrt[b]{x^a} = x^{a/b}$

basic rules

$$\frac{d}{dx}(c) = 0 \qquad \frac{d}{dx}(x^n) = nx^{n-1}$$

$$\frac{d}{dx}(x) = 1 \qquad \frac{d}{dx}(e^x) = e^x$$

product rule

$$\frac{d}{dx}\left(f(x)\cdot g(x)\right) = f'(x)\cdot g(x) + f(x)\cdot g'(x)$$

quotient rule

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\cdot f'(x) - f(x)\cdot g'(x)}{g(x)^2}$$

chain rule

$$\frac{d}{dx}\left(f(g(x))\right) = f'(g(x))\cdot g'(x)$$

exponential (base a)

$$\frac{d}{dx}\left(a^{g(x)}\right) = a^{g(x)}\cdot g'(x)\cdot \ln(a)$$

log (base a)

$$\frac{d}{dx}\left(\log_a(g(x))\right) = \frac{1}{g(x)\cdot \ln(a)}\cdot g'(x)$$

natural log

$$\frac{d}{dx}\left(\ln(g(x))\right) = \frac{1}{g(x)}\cdot g'(x)$$

implicit differentiation

- use when you cant isolate y -

1. differentiate both sides with respect to x

$$\frac{d}{dx}(LHS) = \frac{d}{dx}(RHS)$$

2. solve for $\frac{dy}{dx}$ (or y')

notes

- $\frac{d}{dx}(y) = \frac{dy}{dx}$ (or y')
- $\frac{d}{dx}(y^2) = 2y\frac{dy}{dx}$ (or $2yy'$)
- evaluate $\frac{dy}{dx}$ at (x,y): $\left.\frac{dy}{dx}\right|_{(x,y)}$

logarithmic differentiation

- use when you have a variable in your BASE and EXPONENT $y = g(x)^{f(x)}$ -

1. take ln of <u>both</u> sides

$$\ln(y) = \ln\left(g(x)^{f(x)}\right)$$

2. use law of logs to simplify
3. take derivative of <u>both</u> sides
4. isolate $\frac{dy}{dx}$ (or y')
5. substitute original function, y, into result

notes

- $\frac{d}{dx}(\ln(y)) = \frac{1}{y}\cdot\frac{dy}{dx}$

trig derivatives

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(\cos x) = -\sin x \qquad \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

inverse trig derivatives

$$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\csc^{-1}x) = \frac{-1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\cos^{-1}x) = \frac{-1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\sec^{-1}x) = \frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}(\cot^{-1}x) = \frac{-1}{1+x^2}$$

ya better not forget chain rule!

sin·cos deriv cycle

$\sin x \rightarrow \cos x \rightarrow -\sin x \rightarrow -\cos x \rightarrow \sin x$

special limits

$$\lim_{x\to 0}\frac{\sin x}{x} = 1 \qquad \lim_{x\to 0}\frac{\cos x - 1}{x} = 0$$

- make sure angle & coefficient match! -

derivative applications

Physics

$\frac{d}{dt}$ position $s(t)$

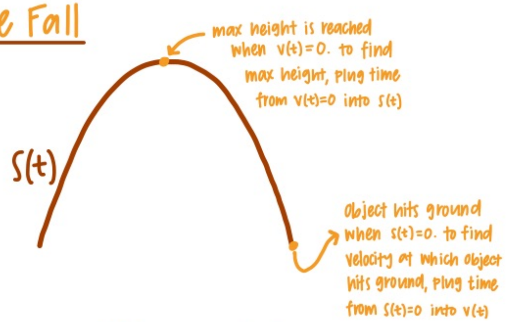
$\frac{d}{dt}$ velocity $v(t)$ (speed $= |v(t)|$)

acceleration $a(t)$

speeding up: $v(t)$ & $a(t)$ same sign

slowing down: $v(t)$ & $a(t)$ diff signs

Free Fall



economics

$C(x)$: total cost to produce x units

average cost: $\frac{C(x)}{x}$

additional cost: $\Delta C = C(x_2) - C(x_1)$

marginal cost: $C'(n) \approx C(n+1) - C(n)$

related rates

1. draw a diagram
2. write down given info & rate you want to find
3. write equation relating variables
4. use implicit differentiation to take derivative with respect to time

   i.e. $\frac{d}{dt}(A) = \frac{d}{dt}(\pi r^2)$
5. plug in given info
6. solve for rate in question

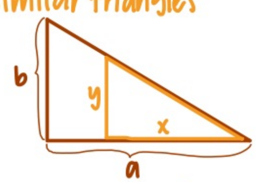
notes

- know basic formulas

  circle area: $A = \pi r^2$

  rectangle area: $A = l \cdot w$

  box volume: $V = l \cdot w \cdot h$
- similar triangles



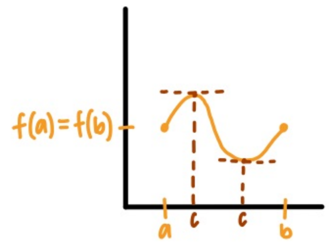
$$\frac{a}{b} = \frac{x}{y}$$

Rolle's Theorem

If $f(x)$ is

- continuous on $[a,b]$
- differentiable on (a,b)
- $f(a) = f(b)$

there is a number c in (a,b) such that $f'(c) = 0$

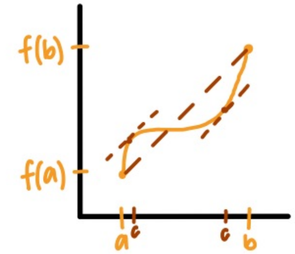


mean value theorem

If $f(x)$ is

- continuous on $[a,b]$
- differentiable on (a,b)

there is a number c in (a,b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$



derivatives tell us...

absolute max/min

Extreme Value Theorem: A continuous function on $[a,b]$ has an abs min & abs max on $[a,b]$

Critical Points: where $f'(x) = 0$ OR $f'(x)$ DNE

Closed Interval Method (find abs min/max):

1. find critical points of f
2. evaluate $f(x)$ at endpoints & critical points (T chart)
3. largest $f(x)$ value = abs max, smallest $f(x)$ value = abs min

x	$f(x)$
a	$f(a)$
cp	$f(cp)$
b	$f(b)$

absolute max/min

Extreme Value Theorem: A continuous function on $[a,b]$ has an abs min & abs max on $[a,b]$

Critical Points: where $f'(x)=0$ OR $f'(x)$ DNE

Closed Interval Method (find abs min/max):
1. find critical points of f
2. evaluate $f(x)$ at endpoints & critical points (T chart)
3. largest $f(x)$ value = abs max, smallest $f(x)$ value = abs min

x	f(x)
a	f(a)
cp	f(cp)
b	f(b)

1^{st} derivatives

- $f'(x) > 0$: $f(x)$ increasing
- $f'(x) < 0$: $f(x)$ decreasing
- local min: decr $\longrightarrow$ incr
- local max: incr $\longrightarrow$ decr

incr/decr test:
1. find where $f'(x)=0$ or $f'(x)$ DNE
2. make a sign chart
3. $f' > 0$: f increasing
   $f' < 0$: f decreasing

CP CP $f'(x)$

plug values in interval into $f'(x)$

2^{nd} derivatives

- $f''(x) > 0$: $f(x)$ CCU $\cup$
- $f''(x) < 0$: $f(x)$ CCD $\cap$
- local min: CCU
- local max: CCD

concavity test:
1. find where $f''(x)=0$ or $f''(x)$ DNE
2. make a sign chart
3. $f'' > 0$: f CCU
   -$'' < 0$: f CCD

$f''(x)$

plug values in interval into $f''(x)$

inflection point: occurs where concavity CHANGES

Sketching functions

1. find domain
2. find intercepts
   - y-int: set $x=0$, solve for y
   - x-int: set $y=0$, solve for x
3. Check for symmetry
   - even: y-axis if $f(-x)=f(x)$
   - odd: origin if $f(-x)=-f(x)$
4. asymptotes
   - HA: $\lim_{x\to\infty} f(x)=L$ or $\lim_{x\to-\infty} f(x)=L$
   - VA: denominator $=0$ & $\lim_{x\to a^-} f(x)=\pm\infty$ OR $\lim_{x\to a^+} f(x)=\pm\infty$
   - slant: long polynomial division
5. find where $f(x)$ incr/decr
6. find local extrema
7. find concavity & inflection points
8. Sketch graph!

Linearization

$L(x)=f(a)+f'(a)(x-a)$

The quantity you want to approximate is put into $L(x)$

$f''(a) < 0 \Rightarrow$ CCD $\Rightarrow$ overestimate

$f''(a) > 0 \Rightarrow$ CCU $\Rightarrow$ underestimate

Differentials

$dy = f'(a)dx$

small change in output — small change in input

L'Hopital's Rule

- $0/0$, ∞/∞
  1. use L'Hopital's Rule
- $\infty-\infty$, $\infty\cdot 0$
  1. use algebra to rewrite as a fraction
  2. use L'Hopital's Rule
- 1^∞, 0^0, ∞^0

  rewrite function $f(x)=e^{\ln(f(x))}$
  1. evaluate $\lim_{x\to a} \ln(f(x))=L$
  2. exponentiate step 1 $\lim_{x\to a} f(x)=e^L$

Optimization

1. objective function: min/max/small/large
   constraints: #
2. use constraints to rewrite obj. function in terms of 1 variable
3. take derivative of new obj. function set $=0$, solve
4. verify using one of the following:

- Closed Interval Method
  plug endpoints & conjectured value into obj. function
  smallest/largest value is abs min/abs max
- first derivative test
  draw sign chart for first derivative
  determine incr/decr
  decr $\to$ incr: $\cup$ abs min
  incr $\to$ decr: $\cap$ abs max
- second derivative test
  find second deriv of obj.
  determine sign of f''
  $f'' > 0 \Rightarrow$ CCU $\Rightarrow$ abs min
  $f'' < 0 \Rightarrow$ CCD $\Rightarrow$ abs max

Antiderivatives

$$\int \underbrace{f(x)}_{\text{integrand}}\, \underbrace{dx}_{\text{variable of integration}} = F(x) + \underbrace{C}_{\text{constant of integration}}$$

you can always check your work!

$$\frac{d}{dx}\left(F(x)+C\right) = f(x)$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$$

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

If you don't know how to find the integral, ask yourself, "What gives me my integrand as an answer to a derivative of a function?"

Initial Value Problems

Given $f'(x)$ and $f(x_1) = y_1$

1. find antiderivative
   $\int f'(x)dx = f(x) + C$
2. use given initial value to solve for C
   $f(x_1) + C = y_1$
3. write specific function
   (antideriv with found value of C)

Physics

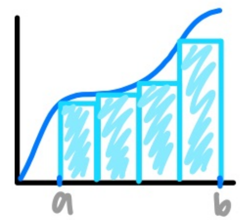
acceleration
$\downarrow$ antiderive
velocity
$\downarrow$ antiderive
position

- feet: accel $= -32\ ft/s^2$
- meters: accel $= -9.8\ m/s^2$
- initial conditions
  - $v(0)$: initial velocity
  - $s(0)$: initial position

Summation Notation

$$\sum_{\underset{\text{start}}{k=1}}^{\overset{\text{stop}}{n}} \underbrace{a_k}_{\text{function}} = a_1 + a_2 + \ldots + a_n$$

ex: $\sum_{k=1}^{4} \frac{k}{2} = \frac{1}{2} + \frac{2}{2} + \frac{3}{2} + \frac{4}{2}$



We can approximate the area under a curve by drawing rectangles under the curve, finding the area for each rectangle, & adding areas together.

width of rectangle: $\Delta x = \frac{b-a}{n}$

height of rectangle: $f(x_k^*)$

- left endpoint: $x_k^* = a + (k-1)\Delta x$
- right endpoint: $x_k^* = a + k\Delta x$
- midpoint: $x_k^* = a + (k - 1/2)\Delta x$

Riemann Sum

$$\sum_{k=1}^{n} \underbrace{f(x_k^*)}_{\text{height}} \underbrace{\Delta x}_{\text{width}}$$

Sum Formulas (don't need to memorize)

$$\sum_{k=1}^{n} c = cn \qquad \sum_{k=1}^{n} k = \frac{n(n+1)}{2} \qquad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

To evaluate Riemann sums, use algebra to get your summation to look like the sum formulas given.

Definite Integrals

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x = \text{net area}$$

(area above − area below)

- $\int_a^a f(x)dx = 0$
- $\int_a^b f(x)dx = -\int_b^a f(x)dx$
- $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

Fundamental Theorem of Calculus (Part I)

$$\frac{d}{dx}\int_a^{g(x)} f(t)\,dt = f(g(x))\cdot g'(x)$$

- function has to be top limit of integration!

Fundamental Theorem of Calculus (Part II)

$$\int_a^b f(x)dx = F(b) - F(a)$$

working with integrals

even·odd functions

Recall: • even: $f(-x)=f(x)$ symmetric about y-axis
• odd function: $f(-x)=-f(x)$ symmetric about origin

if f is even, $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

if f is odd, $\int_{-a}^{a} f(x)\,dx = 0$

average value of a function

$$\bar{f} = \underbrace{\frac{1}{b-a}}_{\text{length of interval}} \underbrace{\int_a^b f(x)\,dx}_{\text{integral}}$$

MVT for integrals: Let f be continuous on [a,b].
There is a point c in (a,b) such that

$$f(c) = \underbrace{\frac{1}{b-a}\int_a^b f(x)\,dx}_{\text{average value of a function}}$$

u-substitution

indefinite integrals

1. identify u where its derivate, du, or a constant multiple of its derivative is in the integrand
2. substitute u and du into integral
3. evaluate integral w/ respect to u. (don't forget +C)
4. replace u w/ function of x so final answer is in terms of x

$$\int f(g(x))\cdot g'(x)\,dx = \int f(u)\,du$$

$u=g(x)$
$du=g'(x)\,dx$

definite integrals

1. identify u where its derivate, du, or a constant multiple of its derivative is in the integrand
2. change bounds of integration
   new lower bound: $u(a)$
   new upper bound: $u(b)$
3. substitute $u, du, u(a)$, and $u(b)$ into integral.
4. evaluate integral w/ respect to u.
   (note: you do not have to make any substitutions to get into terms of x since you have changed your bounds)

$$\int_a^b f(g(x))\cdot g'(x)\,dx = \int_{u(a)}^{u(b)} f(u)\,du$$

$u=g(x)$
$du=g'(x)\,dx$
new lower bound: $u(a)$
new upper bound: $u(b)$